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On the Fock quantisation of the hydrogen atom

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Abstract. In a celebrated work, Fock explained the degeneracy of the energy levels of the Kepler problem (or hydrogen atom) in terms of the dynamical symmetry group SO(4). Making a stereographic projection in the momentum space and rescaling the momenta with the eigenvalues of the energy, he showed that the problem is equivalent to the geodesic flow on the sphere S^3. In this way, the 'hidden' symmetry SO(4) is made manifest. About thirty five years later, Souriau and Moser found a similar result for the *classical* Kepler problem. The present author has shown that the classical n-dimensional Kepler problem can be better understood by enlarging the phase space of the geodesical motion on S^n and including time and energy as canonical variables: a following symplectomorphism transforms the motion on S^n in the Kepler problem. We want to prove in this paper that the Fock procedure is the implementation at 'quantum' level of the above-mentioned symplectomorphism. The interest is not restricted to the old Kepler problem: more recently two other systems exhibiting the same symmetries have been found. They are the McIntosh-Cisneros-Zwanziger system and the geodesic motion in Euclidean Taub-NUT space. Both have a physical interest: they indeed describe a spinless test particle moving outside the core of a self-dual monopole and the asymptotic scattering of two self-dual monopoles, respectively.

1. Introduction

In a celebrated work, Fock [1] explained the degeneracy of the energy levels of the Kepler problem (KP) (or hydrogen atom) in terms of the dynamical symmetry group SO(4). Making a stereographic projection in the momentum space and rescaling the momenta with the eigenvalues of the energy, he showed that the problem is equivalent to the geodesic flow on the sphere S^3. In this way, the 'hidden' symmetry SO(4) is made manifest. About thirty five years later, Souriau [2] and Moser [3] found a similar result for the *classical* KP. Comparing the quantum and the classical treatments, one must face the usual well known problem of the existence of striking similarities joined with subtle differences. The aim of this paper is to make the 'quantisation' of the classical system a clear and mathematically well defined process. The interest is not restricted to the old KP: more recently, two other systems exhibiting the same symmetries have been found. They are the McIntosh-Cisneros-Zwanziger (MICZ) system [4] and the geodesic motion in Euclidean Taub-NUT space [5]. Both have a physical interest: they describe indeed a spinless test particle moving outside the core of a self-dual monopole [6] and the asymptotic scattering of two self-dual monopoles [5], respectively.

We start from some previous results of the author. In [7] we have shown that the classical n-dimensional KP can be better understood by enlarging the phase space of the geodesical motion on S^n and including time and energy as canonical variables: a following symplectomorphism $\mathscr{C}$ sends the motion on S^n in the KP. In [8] we have shown that the motion on S^n can be quantised (in a well defined mathematical sense) thanks to the theory of the representation of semisimple non-compact Lie groups. We want to prove in this paper that the Fock procedure is the implementation at the 'quantum' level of the above-mentioned symplectomorphism $\mathscr{C}$.

In § 2 we briefly review all these previous results, mainly to fix the ideas and the notation. In § 3 we implement the first factor of $\mathscr{C}$ (in fact $\mathscr{C}$ is viewed as the product of three symplectomorphisms), consisting of the interchange of the canonical coordinates: the difficulty is that the corresponding Fourier transform cannot be directly applied, since we have a function on S^n rather than on $\boldsymbol{R}^n$. In § 4 we complete the process, obtaining the quantisation of the various physical systems.

As for the notation, the range of the indices is

$$\begin{aligned} &A, B, C = -1, 0 \ldots n+1 \\ &\mu, \nu, \rho = 0 \ldots n \\ &\alpha, \beta, \gamma = 1 \ldots n+1 \\ &i, j, k = 1 \ldots n. \end{aligned}$$

In the three-dimensional case we use the vector notation.

2. A review of previous results

2.1. *The quantum problem [1]*

Let us consider the Schrödinger equation for the n-dimensional KP:

$$[-\tfrac{1}{2}\Delta - 1/q]\phi(q) = E\phi(q) \tag{2.1}$$

where q_k are cartesian coordinates in $\boldsymbol{R}^n$ and $q = (\Sigma_k q_k^2)^{1/2}$. Performing a Fourier transformation $\mathscr{F}$, write the equation in momentum space. Since [9, p 192]

$$\mathscr{F}(q^\lambda) = 2^{\lambda+n}\pi^{n/2}\frac{\Gamma((n+1)/2)}{\Gamma(-\lambda/2)}p^{-\lambda-n} \tag{2.2}$$

and

$$\mathscr{F}(fg) = (2\pi)^{-n}\mathscr{F}(f)*\mathscr{F}(g) \tag{2.3}$$

(where $f*g$ is the convolution integral) we find

$$\tfrac{1}{2}(p^2 - 2E)\phi(p) = \frac{\Gamma((n-1)/2)}{2\pi^{(n+1)/2}}\int \mathrm{d}^n p' \frac{\phi(p')}{|p-p'|^{n-1}}. \tag{2.4}$$

Let us consider bound states, $E<0$. Now replace p_k by $(-2E)^{1/2}x_k$, then imbed the n-dimensional space into an $(n+1)$-dimensional one with cartesian coordinates X_α and perform a stereographic projection on the unit sphere. The x become local coordinates on S^n (the north pole is missing). Let $\hat{X}_\alpha$ be the coordinates in $\boldsymbol{R}^{n+1}$ of the point with local coordinates x_k on S^n; thus

$$\begin{aligned} \hat{X}_k &= \frac{2(-2E)^{1/2}}{x^2-2E}x_k \\ \hat{X}_{n+1} &= \frac{x^2+2E}{x^2+2E}. \end{aligned} \tag{2.5}$$

An immediate calculation shows that

$$|\hat{X}-\hat{X}'|^2=\frac{-8E}{(x^2-2E)(x'^2-2E)}|x-x'|^2 \tag{2.6}$$

and that the volume element on S^n is

$$\mathrm{d}\Omega=\left(\frac{2(-2E)^{1/2}}{x^2-2E}\right)^n \mathrm{d}^n x. \tag{2.7}$$

Let us also change the wavefunction by defining

$$\Phi(\hat{X})=[(-2E)^{1/2}]^{n/2}\left[\frac{x^2-2E}{-4E}\right]^{(n+1)/2}\phi((-2E)^{1/2}x). \tag{2.8}$$

Inserting in (2.4) we get

$$\frac{\Gamma((n-1)/2)}{2\pi^{(n+1)/2}}\int\frac{\Phi(\hat{X})\,\mathrm{d}\Omega'}{|\hat{X}-\hat{X}'|^{n-1}}=(-2E)^{1/2}\Phi(\hat{X}). \tag{2.9}$$

In this way the $\mathrm{SO}(n+1)$ symmetry is made manifest. In order to solve (2.9), remember the well known (and easy to check) result of the theory of homogeneous symmetric integral equations of the type

$$\int K(X,X')\Phi(X')\,\mathrm{d}X'=\lambda\Phi(X). \tag{2.10}$$

If one can find an orthonormal expansion of the kernel as

$$K(X,X')=\sum_k \lambda_k f_k^*(X')f_k(X)$$

then the eigenfunctions and the eigenvalues of (2.10) are $f_k(X)$ and λ_k. The eigenfunctions of (2.9) are thus the n-dimensional spherical harmonics (i.e. the harmonic homogeneous polynomials in $\boldsymbol{R}^{n+1}$ restricted to S^n) and the eigenvalues are

$$\frac{1}{(-2E)^{1/2}}=l+\frac{n-1}{2}\qquad l=0,1\ldots \tag{2.11}$$

with multiplicity $(2l+n-1)(l+n-2)!/l!(n-1)!$.

2.2. *The classical problem [2, 3, 7]*

The classical KP has the Hamiltonian

$$H=\tfrac{1}{2}p^2-1/q \tag{2.12}$$

where q_k and p_h are canonical coordinates in $T^*(\boldsymbol{R}^n-\{0\})$.

Let $\eta_{AB}=\mathrm{diag}(--+\ldots+)$ be the metric tensor of $\boldsymbol{R}^{2,n+1}$ and $m_{AB}=-m_{BA}$ a basis of the Lie algebra $\mathscr{G}$ of $\mathrm{G}=\mathrm{SO}(2,n+1)$. Then

$$[m_{AB},m_{AC}]=\eta_{AA}m_{BC} \tag{2.13}$$

or zero if all indices are different. It is convenient to introduce special symbols for the elements of the basis, namely

$$\begin{aligned} m_{\mu\nu}&=J_{\mu\nu}\\ m_{\mu n+1}&=A_\mu\\ m_{-1\mu}&=B_\mu\\ m_{-1n+1}&=D \end{aligned} \tag{2.14a}$$

or alternatively

$$P_\mu = A_\mu + B_\mu \qquad C_\mu = A_\mu - B_\mu. \tag{2.14b}$$

We now recall some well known facts [10, 11]. Since the action of G on $\boldsymbol{R}^{2,n+1}$ is linear, it induces an action on the projective manifold of rays through the origin. Moreover G sends the null cone into itself and acts transitively on the manifold M of null rays. This manifold is diffeomorphic to $S^1 \times S^n$ and is endowed with a class of pseudoRiemannian metrics g_γ obtained by restriction of the SO$(2, n+1)$ invariant metric η on any section γ of the null cone. The action of G on M is conformal; the metrics g_γ being conformally flat, with signature $(-+\ldots+)$, the Lie algebra $\mathscr{G}$ of G is isomorphic to the Lie algebra of conformal vector fields on Minkowski space $\boldsymbol{R}^{1,n}$. So we can identify the generators in (2.14*a*, *b*) as follows: $J_{\mu\nu}$ = Lorentz group, D = dilation, P_μ = translations, C_μ = conformal translations. Let H be the (closed) subgroup of G with Lie algebra $\mathscr{H} = \{J_{\mu\nu}, C_\mu, D\}$: it is the isotropy group of the origin in $\boldsymbol{R}^{1,n}$. Since $M = \mathrm{G/H}$, we can identify M with the 'conformal compactification' of $\boldsymbol{R}^{1,n}$. In other words, one can obtain M by adding to $\boldsymbol{R}^{1,n}$ a null cone at infinity.

Let us now consider the symplectic action of G on $T^*(\mathrm{G/H})$. This action not being transitive, we may decompose $T^*(\mathrm{G/H})$ into orbits of G. They are symplectic manifolds on which the group action is transitive, and so they may be identified (Kostant-Souriau theorem, [12 p 180]) with (covering spaces of) orbits of G in $\mathscr{G}^*$. The point is that to obtain the Kepler manifold T^+S^n (i.e. the cotangent bundle to the sphere with the null section deleted) we must restrict ourselves to the subbundle of the null non-vanishing covectors [7, 8]. To see this explicitly, consider, for example, for $E<0$, the section γ given by

$$\begin{aligned}
\hat{X}^{-1} &= \cos x^0 \\
\hat{X}^{0} &= \sin x^0 \\
\hat{X}^{k} &= \frac{2x^k}{x^2+1} \\
\hat{X}^{n+1} &= \frac{x^2-1}{x^2+1}
\end{aligned} \tag{2.15a}$$

which with

$$\begin{aligned}
\hat{Y}_{-1} &= -\sin x^0 y_0 \\
\hat{Y}_{0} &= \cos x^0 y_0 \\
\hat{Y}_{k} &= \tfrac{1}{2}(x^2+1)y_k - \langle x, y\rangle x_k \\
\hat{Y}_{n+1} &= \langle x, y\rangle
\end{aligned} \tag{2.15b}$$

give the moment map $T^*M \mapsto \mathscr{G}^*$:

$$m_{AB} = \hat{Y}_A\hat{X}_B - \hat{Y}_B\hat{X}_A. \tag{2.16}$$

The metric g_γ is

$$\|y\|^2 = -y_0^2 + [\tfrac{1}{2}y(x^2+1)]^2. \tag{2.17}$$

The constraint $\|y\|=0$ on T^*M gives a presymplectic manifold but dividing out the kernel of the 2-form, i.e. setting $x^0=0$, we obtain the symplectic manifold T^+S^n. The moment map (2.16) becomes

$$\begin{aligned}
J_{hk} &= y_h x_k - y_k x_h \\
L_{0k} &= \mp y x_k \\
A_0 &= \mp\tfrac{1}{2}y(x^2-1) \\
A_k &= \tfrac{1}{2}(x^2-1)y_k - \langle x, y\rangle x_k \\
B_0 &= \mp\tfrac{1}{2}y(x^2+1) \\
B_k &= \tfrac{1}{2}(x^2+1)y_k - \langle x, y\rangle x_k \\
D &= \langle x, y\rangle
\end{aligned} \tag{2.18a}$$

and

$$\begin{aligned}
P_0 &= \mp y x^2 \\
P_k &= x^2 y_k - 2\langle x, y\rangle x_k \\
C_0 &= \mp y \\
C_k &= -y_k.
\end{aligned} \tag{2.18b}$$

Let us return to the moment map (2.16) and, before reducing it, consider the canonical transformation $\mathscr{C}_1$:

$$q_k = y_0 y_k \tag{2.19a}$$

$$p_k = -x_k/y_0 \tag{2.19b}$$

$$q_0 = -y_0^3(x_0 - \langle x, y\rangle/y_0) \tag{2.19c}$$

$$p_0 = 1/2y_0^2. \tag{2.19d}$$

$\mathscr{C}_1$ may be viewed as the composition of three canonical transformations: (a) that given by interchanging coordinates and momenta; (b) that given by (2.19a, b), equivalent to an 'energy rescaling'; (c) that given by (2.19d). Note that (2.19c) is forced by requiring canonicity. Now the constraint is

$$p_0 + H(q, p) = 0 \tag{2.20}$$

where $H(q,p) = p^2/2 \mp q^{-1}$. This Hamiltonian is a function of B_0 and thus has the same symmetry group $\mathrm{SO}(n+1)$.

2.3. Including a Dirac monopole [13, 14]

In the physical case, i.e. $n=3$, we have the local isomorphism $\mathrm{SO}(2,4) \approx \mathrm{SU}(2,2)$ and this gives rise to an interesting generalisation of the moment map (2.18).

Let $\mathscr{E}$ be a matrix representation of the $\mathrm{U}(2,2)$-invariant Hermitian form. We choose a basis in $\boldsymbol{C}^{2,2}$ such that $\mathscr{E}$ has the form

$$\begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}.$$

$\boldsymbol{C}^{2,2}$ is equipped with a natural symplectic form $\omega = \mathrm{d}\Theta$, where

$$\Theta = \tfrac{1}{2}\mathrm{i}(\psi^\dagger \mathscr{E}\, \mathrm{d}\psi - \mathrm{d}\psi^\dagger \mathscr{E}\psi) \tag{2.21}$$

and $\psi \in \boldsymbol{C}^{2,2}$. The linear action of U(2, 2) on $\boldsymbol{C}^{2,2}$ is manifestly symplectic. The associated moment map: $\boldsymbol{C}^{2,2} \mapsto \mathrm{u}^*(2,2)$ is easily found to be

$$\mathrm{i}\psi\psi^+\mathscr{E} = \mathrm{i}\begin{pmatrix} zw^+ & zz^+ \\ ww^+ & wz^+ \end{pmatrix} \tag{2.22}$$

where we set $\psi = \binom{z}{w}$, $z \neq 0$. The action of the centre U(1) of U(2, 2) is free on $(\boldsymbol{C}^2 - \{0\}) \oplus \boldsymbol{C}^2$ and induces the reduction of any submanifold T_μ of twistors of constant modulus

$$\psi^+\mathscr{E}\psi = \mu \tag{2.23}$$

to $T_\mu/U(1)$. We will prove that the moment map: $T_\mu/\mathrm{U}(1) \mapsto \mathrm{su}^*(2,2)$ is given by

$$\begin{aligned}
&\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{\pi} - \mu\boldsymbol{x}/x \\
&L = \mp x\boldsymbol{\pi} \\
&A_0 = \mp\tfrac{1}{2}x(\pi^2 - 1) \mp \tfrac{1}{2}\mu^2/x \\
&\boldsymbol{A} = \tfrac{1}{2}(\pi^2 - 1)\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{\pi})\boldsymbol{\pi} + \frac{\mu}{x}\boldsymbol{J} + \frac{1}{2}\frac{\mu^2}{x^2}\boldsymbol{x} \\
&B_0 = \mp\tfrac{1}{2}x(\pi^2 + 1) \mp \tfrac{1}{2}\mu^2/x \\
&\boldsymbol{B} = \tfrac{1}{2}(\pi^2 + 1)\boldsymbol{x} - (\boldsymbol{x}\cdot\boldsymbol{\pi})\boldsymbol{\pi} + \frac{\mu}{x}\boldsymbol{J} + \frac{1}{2}\frac{\mu^2}{x^2}\boldsymbol{x} \\
&D = \boldsymbol{x}\cdot\boldsymbol{\pi}
\end{aligned} \tag{2.24}$$

where now the π_h are not canonical coordinates but satisfy rather

$$\{\pi_h, \pi_k\} = \mu\varepsilon_{hki}x_i/x^3. \tag{2.25}$$

To this purpose, let M be $\boldsymbol{C}^2 - \{0\} = \boldsymbol{R}^4 - \{0\}$ and (z, w) coordinates on $(\boldsymbol{C}^2 - \{0\}) \oplus \boldsymbol{C}^2 = T^*M$. We have the action of U(1) on M given by

$$z \mapsto z\exp(\mathrm{i}\beta/2) \tag{2.26}$$

and we can apply the reduction theorem of Kummer [15].

Theorem 2.1. Let G be a one-parameter Lie group acting freely and properly on M. Let $M \mapsto N = M/G$ be the induced principal fibre bundle and α a connection 1-form on it. The reduced manifold is symplectomorphic to T^*N endowed with a symplectic form given by the canonical one plus a 'magnetic term' $\mu\tau_N^*\,\mathrm{d}\alpha$ (where τ_N is the canonical projection $T^*N \mapsto N$).

Regard M as $\boldsymbol{R}^+ \times S^3$ so that the U(1) action on M gives an induced action on S^3; its quotient is $S^2 = \boldsymbol{CP}^1$. Thus $N = \boldsymbol{CP}^1 \times \boldsymbol{R}^+$. As is well known, this principal U(1) bundle $M \mapsto N$ has a natural connection 1-form α given by

$$\alpha = \frac{\mathrm{Im}(z^+\,\mathrm{d}z)}{z^+z}. \tag{2.27}$$

When restricted to S^2 (i.e. $z^\dagger z=1$), α is the Kähler 1-form on $\boldsymbol{CP}^1$. The action of U(1) commutes with that of SU(2, 2) and thus its Hamiltonian is constant. Parametrise z in terms of the spherical coordinates (x, θ, ϕ) on $\boldsymbol{R}^3-\{0\}$, getting

$$z=\begin{pmatrix} \sqrt{x}\cos\frac{\theta}{2}\exp\left(\mathrm{i}\frac{\phi+\beta}{2}\right) \\ \sqrt{x}\sin\frac{\theta}{2}\exp\left(\mathrm{i}\frac{-\phi+\beta}{2}\right) \end{pmatrix}. \tag{2.28}$$

The angle β is an 'ignorable' coordinate for all the Hamiltonians of the SU(2, 2) action and therefore the conjugate momentum is a constant ($=\mu$). It is easy to verify that the moment map $T^*(\boldsymbol{R}^3-\{0\})\mapsto \mathrm{su}^*(2,2)$, given composing the lift of (2.28) with (2.22), gives just (2.24), as required. The double sign arises since we can choose indifferently $\mp m_{A_{n+1}}$ in the basis of su*(2, 2). The moment map (2.18) can be obtained by (2.24), putting $\mu=0$ and interchanging coordinates and momenta.

Analogously to the preceding case and applying another canonical transformation $\mathscr{C}_2$ (which differs from $\mathscr{C}_1$ since the interchange between coordinates and momenta is missing), we find (see [16] for details) the Hamiltonian of the MICZ system:

$$H=\tfrac{1}{2}p^2\mp 1/q+\tfrac{1}{2}\mu^2/q^2. \tag{2.29}$$

Finally the canonical transformation $\mathscr{C}_3$:

$$\begin{aligned} \boldsymbol{x}&=\left[\left(\frac{\mu}{4m}\right)^2-2p_0\right]^{1/2}\boldsymbol{q} \\ \boldsymbol{\pi}&=\boldsymbol{p}\left[\left(\frac{\mu}{4m}\right)^2-2p_0\right]^{-1/2} \\ x_0&=-\frac{1}{4mp_0}\left[\left(\frac{\mu}{4m}\right)^2-2p_0\right]^{1/2}\left\{\boldsymbol{q}\cdot\boldsymbol{p}+\left[\left(\frac{\mu}{4m}\right)^2-2p_0\right]q_0\right\} \\ y_0&=4m\left(p_0-\left(\frac{\mu}{4m}\right)^2\right)\left[\left(\frac{\mu}{4m}\right)^2-2p_0\right]^{-1/2} \end{aligned} \tag{2.30}$$

give rise to the reduced Hamiltonian of a particle in an Euclidean Taub-NUT space:

$$H=\frac{1}{2}\frac{p^2}{1\mp 4m/q}+\frac{1}{2}\left(1\mp\frac{4m}{q}\right)\left(\frac{\mu}{4m}\right)^2. \tag{2.31}$$

2.4. Quantisation of the Kepler manifold

Now we want to quantise the moment map (2.18), i.e. to construct operators acting over functions on the sphere S^n that close under commutation brackets to give the Lie algebra so(2, $n+1$). We start with the representation of SO(2, $n+1$) through operators acting on $S^1\times S^n$: it is easy to find this representation since this manifold is a homogeneous space for the group. We find the induced representation R_λ:

$$U_{AB}=v^\mu_{AB}\partial_\mu+\left(\frac{n+1}{2}+\lambda\right)\tau_{AB}. \tag{2.32}$$

The v_{AB} are the vector fields generated by the action of SO(2, $n+1$) on $S^1\times S^n$, τ_{AB} the infinitesimal multiplier (null for the isometry subgroup SO(2)$\otimes$SO($n+1$)) and λ

a complex parameter. For an imaginary λ (the so-called principal series) the representation is unitary, whereas for a real λ (the so-called supplementary series) it is unitary only when restricted to the isometry subgroup.

As a basis for the representation space, we may choose functions of the type

$$\phi_{lm} := h_l(\hat{X})\mathrm{e}^{\mathrm{i}mx^0} \tag{2.33}$$

with $l \in \boldsymbol{Z}_+$ and

$$m \in \begin{cases} \frac{1}{2}\boldsymbol{Z} & n \text{ even} \\ \boldsymbol{Z} & n \text{ odd} \end{cases}.$$

Here h_l is a harmonic homogeneous polynomial (HHP) of degree l, i.e.

$$\frac{\partial}{\partial X^\alpha}\frac{\partial}{\partial X_\alpha} h_l(X) = 0 \qquad \text{(harmonicity)} \tag{2.34a}$$

$$X^\alpha \frac{\partial}{\partial X^\alpha} h_l(X) = l h_l(X) \qquad \text{(homogeneity)}. \tag{2.34b}$$

It is well known that a HHP of degree l restricted to S^n gives a spherical harmonic of the same degree. Equation (2.32) gives

$$U_{-10} = -\partial_0 \tag{2.35a}$$

$$U_{\alpha\beta} = X_\beta \frac{\partial}{\partial X^\alpha} - X_\alpha \frac{\partial}{\partial X^\beta} \tag{2.35b}$$

$$U_\alpha^\pm = -\mathrm{e}^{\pm \mathrm{i}x^0}\left[\frac{\partial}{\partial X^\alpha} \mp mX_\alpha - \left(l + \frac{n+1}{2} + \lambda\right)X_\alpha\right]. \tag{2.35c}$$

Define $\mathscr{F}_{lm}$ as the space of the functions of type (2.33): for every fixed couple (l, m) it is the representation space of the maximal subgroup (= isometry subgroup). Define $\mathscr{F} := \oplus_{lm} \mathscr{F}_{lm}$. We have the following.

Theorem 2.2. For $\lambda = -1$ the two subspaces of $\mathscr{F}$ determined by the pairs (l, m) and $(l, -m)$, with

$$m := l + \frac{n-1}{2} \tag{2.36}$$

are invariant under the action of the representation (2.35) and are manifestly isomorphic to the space of functions on S^n.

In this way we obtain an irreducible representation by means of pseudodifferential operators acting on S^n. In fact, consider, for example, the energy operator $\mathrm{i}U_{-10} = B_0$: the eigenvalues of B_0 acting on $\mathscr{F}_{lm}$ are $(l + (n-1)/2)$ with multiplicity $(2l+n-1) \times (l+n-2)!/l!(n-1)!$ and equal to those of $\{-\Delta_{S^n} + [(n-1)/2]^2\}^{1/2}$ acting on S^n. The use of pseudodifferential operators is obviously due to the fact that our dynamical group does not act effectively on S^n. This theorem is the representation-theoretic analogue of the reduction $T^*(S^1 \times S^n) \mapsto T^+S^n$: in fact, the two subspaces of the theorem are annihilated by the operators

$$\mathrm{i}\frac{\partial}{\partial x^0} \pm \left[-\Delta_{S^n} + \left(\frac{n-1}{2}\right)^2\right]^{1/2}$$

and this is the 'quantum' analogue of the 'classical' constraint $\|y\| = 0$ (see (2.17)).

3. Implementation of the interchange between coordinates and momenta

At this point it is natural to ask: what is the precise relation between the Fock and the classical procedure? The Fock procedure seems to be very closely related to the canonical transformation $\mathscr{C}_1$, but, for example, why must we redefine the wavefunction as in (2.8)? In this section we implement the first part of $\mathscr{C}_1$, that consisting of the interchange between coordinates and momenta. To be more precise, we want to find how the representation on S^n of § 2.4 transforms when, in the moment map (2.18), we interchange coordinates and momenta. We thus want to find the generalisation to the n-dimensional case of the quantisation of the moment map (2.24) but with $\mu = 0$: this quantisation has already been found (with μ generic) by Mack and Todorov [17], but in a way that is specific for the three-dimensional case.

It is well known that to the interchange between canonical coordinates corresponds to a Fourier transform at the quantum level. But we cannot perform this transform directly, since the wavefunction is defined on S^n, rather than on $\boldsymbol{R}^n$. It would be possible to obtain a representation on $\boldsymbol{R}^n$ following a reasoning analogous to that of § 2.4, but this way we meet a difficulty: those generators which result Hermitian are now P_0, P_k, J_{hk}, i.e. the generators of the isometry subgroup of $\boldsymbol{R}^n$. The Hamiltonian B_0 of the KP for $E<0$ will be not Hermitian and this is obviously a disaster. The only way to escape this difficulty is to make the representation *fully* unitary but again we fall into a problem: a general method to obtain this exists only (for what concerns us) for the Lorentz subgroup $SO(1, n+1)$.

The unitarisation of a representation of the supplementary series R_λ is obtained by means of a intertwining operator $\mathscr{I}_\lambda$ (see, for example, [18]). It is defined by the property

$$\mathscr{I}_\lambda R_\lambda = R_{-\lambda} \mathscr{I}_\lambda . \tag{3.1}$$

Finding such an operator is equivalent to defining a new scalar product in the Hilbert space of the representation such that R_λ becomes fully unitary. This is easily verified. In fact, if $a \in SO(1, n+1)$ and $x \in S^n$, the group action is given by

$$T_a \psi(x) = \nu^{-(n/2+\lambda)}(a^{-1}x)\psi(a^{-1}x) \tag{3.2}$$

where ν is the *multiplier* of the representation. The natural scalar product is obviously given by

$$\langle \psi, \phi \rangle = \int_{S^n} \psi^* \phi \, d\Omega \tag{3.3}$$

where $d\Omega$ is the Riemannian volume element of the unit sphere S^n. Put $x' = a^{-1}x$; thus $d\Omega' = \nu^{-n} \, d\Omega$. If λ is purely imaginary, R_λ is unitary; in fact

$$\langle T_a \psi, T_a \phi \rangle = \int_{S^n} \psi^*(x')\phi(x') \, d\Omega' = \langle \psi, \phi \rangle . \tag{3.4}$$

Consider now the case λ real. If ϕ is the wavefunction of R_λ, $\mathscr{I}_\lambda \phi$ will be that of $R_{-\lambda}$ and therefore, defining the new scalar product

$$(\psi, \phi)_\lambda := \langle \psi, \mathscr{I}_\lambda \phi \rangle \tag{3.5}$$

we have that R_λ is fully unitary. Notice that λ cannot be arbitrary if we require the definite-positiveness of (3.5): a detailed analysis shows that must be $|\lambda| < n/2$ [19]. A trivial generalisation to the n-dimensional case of the computation given by Bargmann [20] for one dimension gives the explicit form of $\mathscr{I}_\lambda$ in the so-called compact form:

$$\mathscr{I}_\lambda \phi(\hat{X}) = \int_{S^n} \frac{\phi(\hat{X}')}{|\hat{X} - \hat{X}'|^{n-2\lambda}} \, \mathrm{d}\Omega'. \tag{3.6}$$

Here $\hat{X}$ are the cartesian coordinates in $\boldsymbol{R}^{n+1}$ of a point with local coordinates x on the unit sphere S^n. One can verify (3.6) directly. Let X be the coordinates of a point in $\boldsymbol{R}^{1,n+1}$ (with metric tensor η) belonging to the null cone and put $X = r\hat{X}$. Now define

$$(\psi, \phi)_\lambda = -2 \int_{S^n} \psi^*(x_1) \, \mathrm{d}\Omega_1 \int_{S^n} \left(\frac{\eta(X_1, X_2)}{r_1 r_2} \right)^{\lambda - n/2} \phi(x_2) \, \mathrm{d}\Omega_2. \tag{3.7}$$

Remembering the invariance of $\eta(X_1, X_2)$ under the group action and that $ar = \nu(a, x)r$, one verifies that the representation R_λ is unitary with respect to (3.7). But

$$-2 \frac{\eta(X_1, X_2)}{r_1 r_2} = -2\eta(\hat{X}_1, \hat{X}_2) = \eta(\hat{X}_1 - \hat{X}_2, \hat{X}_1 - \hat{X}_2) = |\hat{X}_1 - \hat{X}_2|^2$$

since the vector $(\hat{X}_1 - \hat{X}_2)$ belongs to the Euclidean subspace $\boldsymbol{R}^{n+1}$. Thus, remembering (3.5), we obtain (3.6).

Let us consider the restriction to the subgroup SO$(1, n+1)$ of the representation on S^n of SO$(2, n+1)$ obtained by only considering the generators J_{hk}, A_h, B_k, D, i.e. excluding the pseudodifferential operators: evidently, it is a representation of the supplementary series with $\lambda = -\frac{1}{2}$. λ being negative, the corresponding intertwining operator is a divergent integral that must be regularised [9, 18]. It is thus convenient to pass, through (3.1), to the equivalent representation with $\lambda = \frac{1}{2}$ and hereafter we will only consider this positive value. Comparing (3.6) with (2.9), and bearing in mind (2.11), *we see that* $\mathscr{I}_{1/2}$ *equals*, modulo a numerical factor, *the inverse of the operator* B_0. This is a fundamental point: $\mathscr{I}_\lambda$ is Hermitian operator with respect to the scalar product (3.3) and therefore also with respect to (3.5). Thus we obtain that the representation on S^n equivalent to that found in § 2.4 is fully unitary with respect to (3.5) with $\lambda = \frac{1}{2}$: in fact, the operators J_{0k} and A_0 will also be Hermitian, since they can be obtained through commutation brackets of Hermitian operators.

Our aim is now to carry the wavefunction of the representation from S^n to $\boldsymbol{R}^n$. Remember [19] that the carrier space of the induced representation R_λ of SO$(1, n+1)$ is the space of the homogeneous functions of degree $-(n/2+\lambda)$ on the null cone in $\boldsymbol{R}^{1,n+1}$, i.e. of the functions defined over the *rays* of the null cone. As a manifold on which the wavefunction is defined we may take a section of the null cone. Sectioning it with a hyperplane parallel to the subspace $\boldsymbol{R}^{n+1}$ we obtain S^n, and with one parallel to some ray a paraboloid that can be identified with $\boldsymbol{R}^n$, since its induced metric is flat. Explicitly we have

(i) for S^n

$$X^0 = 1$$

$$X^k = \frac{2x^k}{x^2+1} \tag{3.8a}$$

$$X^{n+1} = \frac{x^2-1}{x^2+1}$$

(ii) for $\boldsymbol{R}^n$

$$X^0 = \frac{x^2+1}{2}$$

$$X^k = x^k \tag{3.8b}$$

$$X^{n+1} = \frac{x^2-1}{2}.$$

Each X of (ii) may be obtained by the corresponding X of (i) multiplying this by $(x^2+1)/2$, and to each n-tuple $(x_1 \ldots x_n)$ correspond, on the two sections, two points that lie on the same ray. To a wavefunction of the representation $\boldsymbol{R}_\lambda$ defined on S^n there corresponds a homogeneous function of degree $-(n/2+\lambda)$ on the null cone and thus a function on $\boldsymbol{R}^n$: given a function on S^n, the corresponding function on $\boldsymbol{R}^n$ is obtained by multiplying it by $[2/(x^2+1)]^{n/2+\lambda}$.

Summing up, to implement the interchange between canonical coordinates we must (a) multiply the wavefunction on S^n by $[2/(x^2+1)]^{(n+1)/2}$ and (b) perform a Fourier transform. We will now find the implementation of the representation on S^n, beginning with the energy operator B_0. Let us start from the eigenvalue equation $B_0\Phi = \eta\Phi$, that we can write in the form

$$\frac{\Gamma((n-1)/2)}{2\pi^{(n+1)/2}} \int_{S^n} \frac{\Phi(\hat{X}')\,\mathrm{d}\Omega'}{|\hat{X}-\hat{X}'|^{n-1}} = \frac{1}{\eta}\Phi(\hat{X}). \tag{3.9}$$

Performing (a) and (b) is equivalent to following the inverse of the Fock procedure but without energy rescaling. Equation (3.9) becomes (now $y_h = -\mathrm{i}\partial/\partial x_h$)

$$\tfrac{1}{2}x(y^2+1)\phi(x) = \eta\phi(x). \tag{3.10}$$

The implementation of the representation on S^n is

$$\begin{aligned}
J_{hk} &= x_h y_k - x_k y_h \\
L_k &= \mp x y_k \\
A_0 &= \mp\tfrac{1}{2}x(y^2-1) \\
A_k &= \tfrac{1}{2}x_k(y^2-1) - y_k\langle x, y\rangle + \mathrm{i}\frac{n-3}{2}y_k \\
B_0 &= \mp\tfrac{1}{2}x(y^2+1) \\
B_k &= \tfrac{1}{2}x_k(y^2+1) - y_k\langle x, y\rangle + \mathrm{i}\frac{n-3}{2}y_k \\
D &= \langle x, y\rangle - \mathrm{i}\frac{n-1}{2}
\end{aligned} \tag{3.11}$$

where now the ordering of the operators *is* important. The form of B_0 follows directly from (3.10), that of J_{hk} and P_k is obvious and that of the other generators can be determined through brackets. $\mathcal{I}_{1/2}$ is a convolution operator and its Fourier transform will be a multiplication operator: from (3.6) and (2.2) it follows that the representation (3.11) is unitary with respect to the scalar product

$$(\psi, \phi) = \int_{\boldsymbol{R}^n} \psi^*\phi\frac{1}{x}\,\mathrm{d}^n x. \tag{3.12}$$

As in the classical case, the restriction of (3.11) to the three-dimensional case can be generalised to include a Dirac monopole. This representation is obtained in [17] by quantising the moment map $T^*(\boldsymbol{R}^4 - \{0\}) \mapsto \mathrm{sp}^*(8\boldsymbol{R})$ and restricting it to the subgroup U(2, 2): this moment map is quadratic in the coordinates and momenta of $T^*(\boldsymbol{R}^4 - \{0\})$ and thus, as is well known, its quantisation is standard. As in the classical case, the magnetic monopole arises since U(2, 2) has U(1) as a centre.

4. Quantisation of the physical systems

The quantisation of the physical systems, i.e. the KP, MICZ and reduced Taub-NUT systems, now follows quite easily. It has already been given in [21] but in a pure computational way.

For the n-dimensional KP let us consider the eigenvalue equation (3.10) and make the transformation (now $p_h = -\mathrm{i}\partial/\partial q_h$)

$$x_k = q_k/\eta \qquad y_h = \eta p_h. \tag{4.1}$$

We obtain

$$(\tfrac{1}{2}p^2 - 1/q)\phi = \frac{1}{-2\eta^2}\phi \tag{4.2}$$

i.e. the eigenvalue equation of the KP for $E<0$ (for $E>0$ we start with $A_0\phi = \eta\phi$). Obviously the transformation (4.1) and the change in the eigenvalue parameter: $\eta \to -1/2\eta^2$ are equivalent to the canonical transformation $\mathscr{C}_1$. Notice however that, since the rescaling in (4.1) is not made with a constant but with the eigenvalues, the Hilbert space on which the energy operator acts, changes. In fact, the energy operator of (4.2) is Hermitian with respect to the scalar product $\int \psi^*\phi \, \mathrm{d}^n x$.

The quantisation of the other systems follows in a very similar way.

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